Practice:

Let $g(x,y) = 3^x \sec(xy^2)$

1. Find $\partial_x g$
2. Find $\partial_y g$
3. Find $\partial_x \partial_y g$
4. Find $\partial_y \partial_x g$

---

① $\partial_x \left(3^x \sec(xy^2)\right) = (\ln 3) 3^x \sec(xy^2)$
$+ 3^x \sec(xy^2)\tan(xy^2) \cdot y^2$

② $\partial_y \left(3^x \sec(xy^2)\right) = 3^x \sec(xy^2)\tan(xy^2) \cdot 2xy$

③ $\partial_x \partial_y = \dfrac{\partial^2}{\partial_x \partial_y}$

$\partial_x \partial_y \left(3^x \sec(xy^2)\right) = \partial_x \left(3^x \sec(xy^2)\tan(xy^2) \cdot 2xy\right)$

$$\boxed{\left(f(x)g(x)h(x)\cdots\right)' = f'(x)g(x)h(x)\cdots + f(x)g'(x)h(x)\cdots + f(x)g(x)h'(x)\cdots}$$

$= (\ln 3) 3^x \sec(xy^2)\tan(xy^2) \cdot 2xy + 3^x \sec(xy^2)\tan(xy^2) \cdot y^2 \cdot \tan(xy^2) \cdot 2xy$

$+ 3^x \sec(xy^2) \cdot \sec^2(xy^2) \cdot y^2 \cdot 2xy + 3^x \sec(xy^2)\tan(xy^2) \cdot 2y$

④ $\partial_y \partial_x \left(3^x \sec(xy^2)\right) = \partial_y \left((\ln 3) 3^x \sec(xy^2) + 3^x \sec(xy^2)\tan(xy^2) \cdot y^2\right)$

$= (\ln 3) 3^x \sec(xy^2)\tan(xy^2) \cdot 2xy + 3^x \sec(xy^2)\tan(xy^2) \cdot 2xy \tan(xy^2) y^2$

$+ \underline{3^x \sec(xy^2)\sec^2(xy^2)\cdot 2xy\cdot y^2} + \underline{3^x \sec(xy^2)\tan(xy^2)(2y)}.$

Important fact: "Mixed partials commute"

- If a function $f(x,y,\dots)$ is C^k
(ie all the k^{th} partial derivatives of f are continuous fcns)
the k^{th} partial derivatives can be taken in any order and yield the same result.

Example: $f(x,y) = \arctan\left(\frac{x-1}{y}\right)$.

$$\Rightarrow f_y = \frac{1}{1+\left(\frac{x-1}{y}\right)^2}\cdot\left(\frac{x-1}{y}\right)_y$$

$(x-1)y^{-1}$

$$= \frac{1}{1+\left(\frac{x-1}{y}\right)^2}\cdot\frac{-(x-1)}{y^2} = \boxed{\frac{-(x-1)}{y^2+(x-1)^2}}$$

Sagemath.

Type some Sage code below and press Evaluate.

```
f(x,y) = sqrt(x*y^x-1)
show(diff(f(x,y),y))
```

Directional Derivative.

If u is a unit vector

$$u \doteq (u_1, u_2, \dots)$$

$$\sqrt{u_1^2 + u_2^2 + \cdots} = 1 .$$

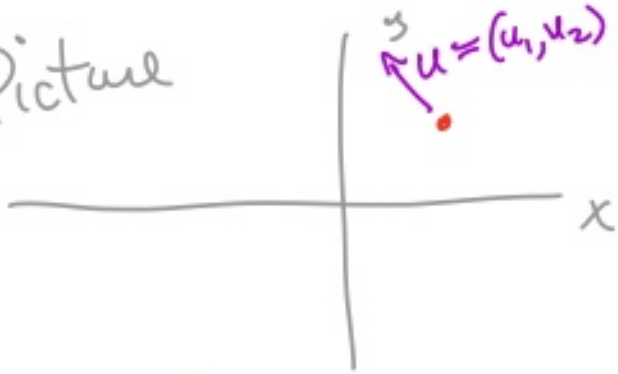
The directional Derivative in direction u

is $\displaystyle (D_u f)(x,y) = \frac{\partial f}{\partial u}(x,y) = \lim_{h \to 0} \frac{f(x+hu_1, y+hu_2) - f(x,y)}{h}$

defn.

= instantaneous rate of change of f as we move in the $(u_1, u_2, \cdots)$ direction.

Picture



Note: $\displaystyle (D_u f)(x,y) = \frac{d}{dh}\Big|_{h=0} \left(f(x+hu_1, y+hu_2) \right)$

chain rule

$$= \frac{\partial f}{\partial x} \frac{\partial (x+hu_1)}{\partial h} + \frac{\partial f}{\partial y} \frac{\partial (y+hu_2)}{\partial h}$$

$$\boxed{D_u f = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2}$$

to calculate

Example: Find the directional derivative of $f(x,y) = x^2 + y^2$ at the point $(1,2)$

(a) in direction of $\langle 1, 1 \rangle$.

$(x,y) = (1,2)$, $u = \frac{1}{\sqrt{2}}\langle 1,1 \rangle$

To make a vector v into a unit vector in the same direction, divide by $|v|$.

$$\Rightarrow D_u f = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$= (\underset{1}{2x}) \frac{1}{\sqrt{2}} + (\underset{2}{2y}) \frac{1}{\sqrt{2}}$$

$$= \frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}} = \frac{6}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \boxed{3\sqrt{2}}$$

(b) in the direction of $\langle -4, 3 \rangle$.

$$u = \frac{1}{\sqrt{25}}\langle -4, 3 \rangle = \left(\frac{-4}{5}, \frac{3}{5}\right)$$

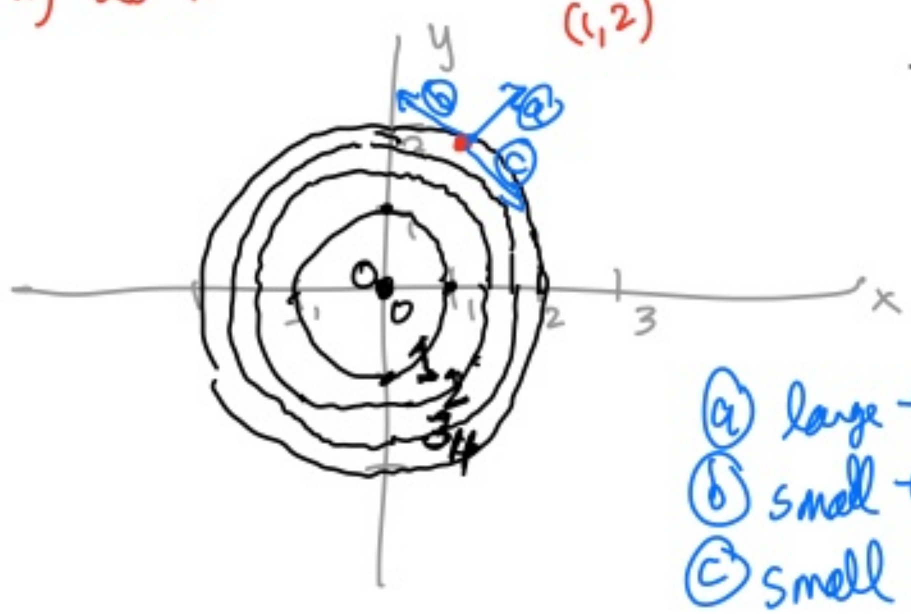
$$D_u f = \frac{\partial f}{\partial x} u_1 + \frac{\partial f}{\partial y} u_2$$

$$= \underset{1}{2x} \cdot \underset{\frac{-4}{5}}{u_1} + \underset{2}{2y}\, \underset{\frac{3}{5}}{u_2} = \frac{-8}{5} + \frac{12}{5} = \boxed{\frac{4}{5}}.$$

(c) in the direction of $\langle 4, -3 \rangle$.

$$D_u f = \boxed{\frac{-4}{5}}$$

Why do these #s make sense?



$f(x,y)=x^2+y^2$

$x^2+y^2=1$

$x^2+y^2=2$

ⓐ large + #.

ⓑ small + #.

ⓒ small ~~neg~~ #.